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COMPUTATION OF NIELSEN NUMBERS FOR CERTAIN MAPS OF HYPERBOLIC SURFACES

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Abstract. Let X be a closed surface for which the Euler characteristic $\chi(X)$ is negative, and let $f : X \to X$ be a self-map that is not surjective. In this short paper, we prove that we can compute the Nielsen number of f, $N(f)$, under some algebraic conditions.

1. Introduction

Let X be a hyperbolic surface, that is, a compact connected surface for which the Euler characteristic $\chi(X)$ is negative and let $f : X \to X$ be a self-map. The Nielsen number of f, $N(f)$, is a homotopy invariant and provides a lower bound for the minimum number of fixed points over all maps homotopic to f. See [1, 5, 8] for the background.

Unfortunately, computing the Nielsen number is difficult and it is particularly difficult on hyperbolic surfaces. See [2, 7, 12] for the details. But recently, for X a hyperbolic surface with boundary, many methods are developed in the papers [3, 4, 9, 10, 13, 14] to compute the Nielsen number $N(f)$.

In this paper, we will first introduce briefly these methods. Then for X a closed hyperbolic surface, we will show that we can apply those methods to compute the Nielsen number $N(f)$ on X. The result in this paper is a partial answer to one of open problems in [7]. The question is that is there an algorithm for the calculation of the Nielsen number for a self-map of a surface?

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2. Nielsen number on hyperbolic surfaces with boundary

Let X be a hyperbolic surface with boundary. Then X is homotopy equivalent to a wedge of a finite number of circles and has fundamental group $\pi_1(X)$ that is a finitely generated free group. Let $\{a_1, \cdots, a_n\}$ be a set of generators of $\pi_1(X)$. Let $f : X \to X$ be a self-map and let $f_\# : \pi_1(X) \to \pi_1(X)$ be the induced endomorphism of f. Since the Nielsen number is a homotopy type invariant [5, p. 21], we may assume that X is a wedge of a finite number of circles if necessary.

Let G be a group and let $\varphi\colon G \to G$ be an endomorphism. Two elements $u, v \in G$ are *Reidemeister equivalent*, also called *twisted conjugate*, if there exists $z \in G$ such that

$$u = \varphi(z)vz^{-1}. \tag{2.1}$$

The challenge in computing $N(f)$ on X is determining whether two elements in $\pi_1(X)$ are Reidemeister equivalent with $\varphi = f_\#$. See [2] for the background.

In 1999, Wagner introduced an algorithm, which is now called Wagner's algorithm, for computing $N(f)$ on X. Wagner's algorithm applies to maps with remnant. For each i, the word $f_\#(a_i)$ *has remnant* if there is a nontrivial subword of $f_\#(a_i)$ which does not cancel in any product of the form

$$f_\#(a_j)^{\pm 1} f_\#(a_i) f_\#(a_k)^{\pm 1}$$

except if j or k equals i and the exponent is -1. The map f *has remnant* if every word $f_\#(a_i)$ has remnant.

THEOREM 2.1 ([13]). *If $f : X \to X$ has remnant, we can compute the Nielsen number $N(f)$ by Wagner's algorithm.*

Wagner's remnant was extended to k-remnant($k \in \mathbb{N}$) in [4]. The remnant condition requires that there is limited cancellation in each product $f_\#(u)f_\#(v)$ when u and v have length 1. Roughly, a map has k-remnant if there is limited cancellation in each product $f_\#(u)f_\#(v)$ when u and v have length k in $\pi_1(X)$.

THEOREM 2.2 ([4]). *If $f : X \to X$ has k-remnant, there is an algorithm for computing the Nielsen number $N(f)$.*

Hart in [2, 3] developed also two other algebraic methods, MRN maps (when $\pi_1(X)$ is free on two generators) and $2C3$ maps, for determining the Nielsen equivalence classes. Roughly, these maps have partial remnant and have some restrictions on the cancellation in the word product among images of generators a_i under $f_\#$.

THEOREM 2.3 ([3]). *For MRN maps and $2C3$ maps, we can determine the Nielsen equivalence classes, so we can compute the Nielsen numbers.*

In the paper [10], Wagner's idea was extended in a different way. The possible lengths of solutions were considered to Equation (2.1). Let F be a finitely generated free group and let $\varphi\colon F \to F$ be an endomorphism. A pair (u, v) of two elements of F *has bounded solution length* (or BSL) if there exists an integer n such that there is no solution $z \in F$ with $|z| > n$ to the equation (2.1)

$$u = \varphi(z)vz^{-1}.$$

The smallest such n is called *the solution bound* (or SB) for (u, v).

Given any pair (u, v) of elements of F, if (u, v) has BSL, we can algorithmically determine whether or not u and v are Reidemeister equivalent by checking for equality of $u = \varphi(z)vz^{-1}$ where z ranges over all elements of F with $|z| \leq SB$. For a map $f : X \to X$, if any pair of two elements in $\pi_1(X)$, each of which represents a fixed point class of f, has BSL for the endomorphism $\varphi = f_\#$ then we say that f *has bounded solution length* (BSL). The maximum of all SB for such pairs is called *the solution bound (SB) for f.*

THEOREM 2.4 ([10]). *If $f : X \to X$ has BSL (and we know the SB for f), then we can algorithmically determine the Nielsen equivalence classes, so we can compute $N(f)$.*

Let X be the pants surface, the 2-sphere with three disjoint open disks removed, or more generally, let X be a compact polyhedron that is homotopy equivalent to the figure-eight. Yi and this author in [11, 14] extended Wagner's work using the concept of the mutant of a map, which had been introduced by Jiang [6], so that an algorithm for computing the Nielsen number on X was completed.

THEOREM 2.5 ([11]). *Let X be an aspherical figure-eight type finite polyhedron and let $f : X \to X$ be a self-map. There is an algorithm for computing the Nielsen number $N(f)$.*

This algorithm is now called the WYK-algorithm.

3. Nielsen number on closed hyperbolic surfaces

Let X be a closed surface of genus $n \geq 2$. Then we have

$$\pi_1(X) = \langle a_1, a_2, \cdots, a_{2n-1}, a_{2n} \mid a_1a_2a_1^{-1}a_2^{-1} \cdots a_{2n-1}a_{2n}a_{2n-1}^{-1}a_{2n}^{-1} \rangle$$

where the relator is the product of n commutators. Let $f : X \to X$ be a self-map and let $f_\# : \pi_1(X) \to \pi_1(X)$ be the induced endomorphism of f. Let F be the free group on the generators $\{a_1, a_2, \cdots, a_{2n}\}$. Given a particular representation of $f_\#$, let $f_{\#F} : F \to F$ be the homomorphism for which $f_\#(a_i)$ and $f_{\#F}(a_i)$ look identical as strings of letters for each generator a_i. The notation $f_{\#F}$ was introduced in [2] and we use the same notation in this section.

THEOREM 3.1. *Let X be a closed hyperbolic surface and let $f : X \to X$ be a self-map that is not surjective. If f satisfies one of the following:*

1. *$f_{\#F}$ has remnant or k-remnant,*
2. *$f_{\#F}$ is a $2C3$ map,*
3. *$f_{\#F}$ has BSL,*

then there is an algorithm for computing the Nielsen number $N(f)$.

Proof. Let X be a closed hyperbolic surface of genus n and let $f : X \to X$ be a self-map that is not surjective. Take a point x in $X - f(X)$. Let A be a regular neighborhood of a wedge of $2n$ circles and identify A with a strong deformation retract of $X - \{x\}$. Then for the subspace A of X, we may consider that the wedge point is the base point of X and that each circle with fixed orientation represents a_i-loop. Since A is a strong deformation retract of $X - \{x\}$, which contains $f(X)$, and the Nielsen number is a homotopy invariant, we may assume that the image of f is into A. For instance, we can retract the image of f into A using a strong deformation retraction of $X - \{x\}$ onto A.

Let $\bar{f} : X \to A$ be the corestriction of f to A. Then

$$f = i \circ \bar{f}$$

where $i : A \to X$ is the inclusion map. Let $f_A : A \to A$ be the map obtained from f by commutation, that is

$$f_A = \bar{f} \circ i.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\bar{f}"'] & X \arrow[d, "\bar{f}"] \\ A \arrow[ur, "i"] \arrow[r, "f_A"'] & A \end{tikzcd}$$

Since the Nielsen number has the commutativity property, we have

$$N(f) = N(f_A).$$

The subspace A of X is a hyperbolic surface with boundary and $\pi_1(A) = F$. For each i, we have

$$(f_A)_\#(a_i) = \bar{f}_\# \circ i_\#(a_i) = \bar{f}_\#(a_i) = f_{\#F}(a_i).$$

Thus we have

$$(f_A)_\# = f_{\#F}.$$

Consequently, if $f_{\#F}$ has remnant or k-remnant (resp. $f_{\#F}$ is a $2C3$ map, $f_{\#F}$ has BSL), then by Theorem 2.1 or Theorem 2.2 (resp. Theorem 2.3, Theorem 2.4), there is an algorithm for computing $N(f_A)$, which equals $N(f)$. □

Since $f_{\#F}(F)$ is a subgroup of the free group F, the group $f_{\#F}(F)$ is also a free group.

THEOREM 3.2. *Let X be a closed hyperbolic surface and let $f : X \to X$ be a self-map that is not surjective. If the rank of the free group $f_{\#F}(F)$ is 2 then there is an algorithm for computing the Nielsen number $N(f)$.*

Proof. Using the same arguments in the proof of Theorem 3.1, we have that $N(f) = N(f_A)$ and $(f_A)_\# = f_{\#F}$, where A and f_A are the same as that in the proof of Theorem 3.1. Thus we will show that there is an algorithm for computing $N(f_A)$.

Let Y be the figure-eight and let F_2 be the fundamental group of Y that is a free group of rank 2. Since $(f_A)_\#(F) = f_{\#F}(F)$ is also a free group of rank 2, the homomorphism $(f_A)_\#$ factors through F_2, that is, there are homomorphisms $\phi : F \to F_2$ and $\psi : F_2 \to F$ such that $(f_A)_\# = \psi \circ \phi$.

$$\begin{tikzcd}
F \arrow[r, "(f_A)_\#"] \arrow[d, "\phi"'] & F \\
F_2 \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Since A and Y are $K(\pi, 1)$-spaces, there are maps $g : A \to Y$ and $h : Y \to A$ such that $g_\# = \phi$, $h_\# = \psi$ and $h \circ g$ is homotopic to f_A so that we have

$$N(f_A) = N(h \circ g).$$

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "g"'] & A \arrow[d, "g"] \\ Y \arrow[ur, "h"] \arrow[r, "f_Y"'] & Y \end{tikzcd}$$

Let $f_Y = g \circ h$ be the commutation of $h \circ g$. Then f_Y is a self-map of Y and since the Nielsen number has the commutativity property, we have

$$N(f_A) = N(f_Y).$$

By Theorem 2.5, there is an algorithm for computing $N(f_Y)$. □

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